

LOCALIZATION OF FLOER HOMOLOGY OF ENGULFABLE TOPOLOGICAL HAMILTONIAN LOOP

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ABSTRACT. Localization of Floer homology is first introduced by Floer [Fl3] in the context of Hamiltonian Floer homology. The author employed the notion in the Lagrangian context for the pair $(\phi_H^1(L), L)$ of compact Lagrangian submanifolds in tame symplectic manifolds (M, ω) in [Oh1, Oh2] for a compact Lagrangian submanifold L and C^2 -small Hamiltonian H . In this article, we extend the localization process for any engulfable Hamiltonian path ϕ_H whose time-one map ϕ_H^1 is sufficiently C^0 -close to the identity (and also to the case of triangle product), and prove that the value of local Lagrangian spectral invariant is the same as that of global one. Such a Hamiltonian path naturally occurs as an approximating sequence [Oh15] of engulfable topological Hamiltonian loop.

We apply this localization to the graphs $\text{Graph } \phi_H^t$ in $(M \times M, \omega \oplus -\omega)$ and localize the Hamiltonian Floer complex of such a Hamiltonian H . This study plays an important role in the proof of homotopy invariance of the spectral invariants of topological Hamiltonian flows proved in [Oh15].

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CONTENTS

1. Introduction and the main results	2
1.1. Topological Hamiltonian loops	3
1.2. Statement of main results	4
2. Local Floer complex for C^2 -small H ; review	8
3. Comparison of two Cauchy-Riemann equations	11
4. Thick-thin decomposition for engulfable C^0 -approximate loop ϕ_H	13
5. Local Floer complex of engulfable C^0 -approximate loop ϕ_H	14
6. Handle sliding lemma for engulfable isotopy of C^0 -approximate loops	16
7. Computation of Local Floer homology $HF_*^{[id]}((L, L), H; U)$	17
8. Localization on the cotangent bundle	19

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9. Appendix: Local Floer complex of engulfable smooth Hamiltonian C^0 -approximate loop	21
9.1. Hamiltonian Floer complex	21
9.2. Isolating local Floer complex	23
9.3. Fix ϕ_G^1 versus $\Delta \cap \text{graph } \phi_G^1$	24
9.4. Localization of triangle product	26
9.5. Localization of the basic phase function	28
References	30

1. INTRODUCTION AND THE MAIN RESULTS

The construction of the local version of the Floer homology was introduced by Floer [Fl3]. The present author applied this construction to the Lagrangian context and defined the local Floer homology, denoted by $HF(H, L; U)$, which singles out the contribution from the Floer trajectories whose images are contained in a given Darboux neighborhood U of L in M . Such an isolation of the contribution is proven to be possible and the resulting Floer homology is isomorphic to the singular homology $H_*(L)$ (with Z_2 -coefficients) in [Oh2], *provided H is C^2 -small*. This C^2 -smallness is used, conspicuously in [Oh2], so that first

$$\phi_{H^t}(L) \subset V \subset \overline{V} \subset U \quad (1.1)$$

holds for all $t \in [0, 1]$, and then the ‘thick-thin’ decomposition of the Floer trajectories exists. The necessity of such a decomposition is highlighted for the Floer moduli spaces for the boundary map, but its necessity is less conspicuous for that of the chain map in [Oh2]:

But this latter was further scrutinized and exploited by Chekanov in his study of displacement energy in [Che1, Che2]. It follows from his argument in [Che1] that the quasi-isomorphism property of thin part of Floer chain maps between the local Floer complex H and the Morse complex of f holds for a sufficiently small $\varepsilon > 0$ as long as $\|H\| < \frac{1}{2}\sigma(M, L, J_0)$ for some ε -regularity type invariant $\sigma(M, L, J_0)$ *as long as the thick-thin decomposition exists for the chain map*. We will recall the definition of $\sigma(M, L, J_0)$ in section 2 for readers’ convenience. The required thick-thin decomposition was established via the thick-thin decomposition of associated Floer moduli spaces into those with big areas and those with very small areas. It was proved in [Oh2] that this dichotomy exists when H is C^2 -small which is in turn proved by a variation of Gromov-Floer compactness as $H \rightarrow 0$ in C^2 -topology. (We would like to emphasize that this convergence argument is not the standard Gromov-Floer type compactness argument since the limiting configuration is degenerate. The precise study of this convergence belongs to the realm of the so called adiabatic limit in the sense of [FO, Oh1, Oh8]. In [Oh2], it was enough to establish a non-constant component in the ‘limit’ which can be proved by a simple convergence argument under an energy bound.)

The main purpose of the present paper is to generalize these constructions by replacing the C^2 -smallness of H (or C^1 -smallness of ϕ_H) by the weaker hypothesis, the C^0 -smallness of the time-one map $\phi_H^1 : t \mapsto \phi_H^t$ for any engulfable Hamiltonian path ϕ_H . One big difference between the C^1 -topology and the C^0 -topology is that C^0 -topology is a priori too weak to control the analytical behavior

of pseudo-holomorphic curves with boundary lying on $\phi_H^1(L)$ in general while C^1 -topology of Lagrangian boundary condition controls analysis of pseudo-holomorphic curves. Because of this, the above mentioned area consideration for the C^2 -small H cannot produce the required thick-thin decomposition.

We will instead use the *maximum principle* to single out ‘thin’ trajectories which turns out to be the best way of obtaining such decomposition even for the C^2 -small H ’s in hindsight.

1.1. Topological Hamiltonian loops. In [OM], Müller and the author introduced the notion of hamiltonian topology on the space

$$\mathcal{P}^{ham}(Symp(M, \omega), id)$$

of Hamiltonian flows $\lambda : [0, 1] \rightarrow Symp(M, \omega)$ with $\lambda(t) = \phi_H^t$ for some time-dependent Hamiltonian H . We first recall the definition of this hamiltonian topology.

Following the notations of [OM], we denote by ϕ_H the Hamiltonian path

$$\phi_H : t \mapsto \phi_H^t; [0, 1] \rightarrow Ham(M, \omega)$$

and by $\text{Dev}(\lambda)$ the associated normalized Hamiltonian

$$\text{Dev}(\lambda) := \underline{H}, \quad \lambda = \phi_H \tag{1.2}$$

where \underline{H} is defined by

$$\underline{H}(t, x) = H(t, x) - \frac{1}{\text{vol}_\omega(M)} \int_M H(t, x) \omega^n. \tag{1.3}$$

Definition 1.1. Let (M, ω) be a closed symplectic manifold. Let λ, μ be smooth Hamiltonian paths. The *hamiltonian topology* of Hamiltonian paths is the metric topology induced by the metric

$$d_{ham}(\lambda, \mu) := \overline{d}(\lambda, \mu) + \text{leng}(\lambda^{-1}\mu). \tag{1.4}$$

Now we recall the notion of topological Hamiltonian flows and Hamiltonian homeomorphisms introduced in [OM].

Definition 1.2 ($L^{(1, \infty)}$ topological Hamiltonian flow). A continuous map $\lambda : \mathbb{R} \rightarrow \text{Homeo}(M)$ is called a topological Hamiltonian flow if there exists a sequence of smooth Hamiltonians $H_i : \mathbb{R} \times M \rightarrow \mathbb{R}$ satisfying the following:

- (1) $\phi_{H_i} \rightarrow \lambda$ locally uniformly on $\mathbb{R} \times M$.
- (2) the sequence H_i is Cauchy in the $L^{(1, \infty)}$ -topology locally in time and so has a limit H_∞ lying in $L^{(1, \infty)}$ on any compact interval $[a, b]$.

We call any such ϕ_{H_i} or H_i an *approximating sequence* of λ . We call a continuous path $\lambda : [a, b] \rightarrow \text{Homeo}(M)$ a *topological Hamiltonian path* if it satisfies the same conditions with \mathbb{R} replaced by $[a, b]$, and the limit $L^{(1, \infty)}$ -function H_∞ called a $L^{(1, \infty)}$ *topological Hamiltonian* or just a *topological Hamiltonian*.

We call a topological Hamiltonian path λ a loop if $\lambda(0) = \lambda(1)$. Any approximating sequence ϕ_{H_i} of a topological Hamiltonian loop λ has the property $\phi_{H_i}^1 \rightarrow id$ in addition to the properties (1), (2) of Definition 1.2.

1.2. Statement of main results. Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold and let $V \subset \overline{V} \subset U$ be a pair of Darboux neighborhoods of L . We denote $\omega = -d\Theta$ on U where Θ is the Liouville one-form on U regarded as an open neighborhood of the zero section of T^*L . Following [Oh8], [Sp], we introduce the following notion. (Similar concept was previously used by Laudenchbach [L] in the context of classical symplectic topology.)

We measure the size of the Darboux neighborhood V by the following constant

$$d(V, \Theta) := \max_{x \in V} |p(x)|, \quad x = (q(x), p(x)). \quad (1.5)$$

This constant is bounded away from 0 and so there exists some $\eta > 0$ depending only on $(V, -d\Theta)$ (and so only on (M, ω)).

Definition 1.3. We call an isotopy of Lagrangian submanifold $\{L_t\}_{0 \leq t \leq 1}$ of L is called V -*engulfable* if there exists a Darboux neighborhood V of L such that $L_s \subset V$ for all s . When we do not specify V , we just call the isotopy engulfable.

We call a (topological) Hamiltonian path ϕ_H engulfable if its graph $\text{Graph } \phi_H^t$ is engulfable in a Darboux neighborhood of the diagonal Δ of $(M \times M, \omega \oplus -\omega)$.

We denote

$$\mathcal{H}_\delta^{\text{engulf}}(L; V) = \{H \mid \phi_H^t(L) \subset V \forall t \in [0, 1], \overline{d}(\phi_H^1, id) \leq \delta\} \quad (1.6)$$

$$\mathfrak{Iso}_\delta^{\text{engulf}}(L; V) = \{L' \in \mathfrak{Iso}(L) \mid L' = \phi_H^1(L), H \in \mathcal{H}_\delta^{\text{engulf}}(L; V)\}. \quad (1.7)$$

The main goal of the present paper is to extend the notion of local Floer homology introduced in [Fl3, Oh2] for the C^2 -small Hamiltonian H to the case of H such that

- (1) its Hamiltonian paths ϕ_H are engulfable,
- (2) its time-one map ϕ_H^1 is C^0 -small.

Such a sequence of smooth Hamiltonian paths naturally occurs as an approximating sequence of engulfable topological Hamiltonian loop (based at the identity). We note that any sufficiently C^2 -small satisfies these properties.

We would like to remark that it is established in [Oh2] that if $\|H\|_{C^2} < C$, then the following automatically hold:

- (1) its Hamiltonian paths ϕ_H is V -engulfable,
- (2) we have a priori action bounds depending only on C for the Hamiltonian chords,
- (3) and the uniform area bounds of the associated connecting Floer trajectories on V , where we regard V as a neighborhood of the zero section in the cotangent bundle so that we use the classical action functional to measure the actions.

In [Oh2], we mainly used the *area* of Floer trajectories to obtain the thick-thin decomposition of the Floer boundary operator $\partial = \partial_0 + \partial'$, which is equivalent to the corresponding dichotomy in terms of filtration changes under the boundary map (or the Floer chain map) for a C^2 -small Hamiltonian H .

However for the Hamiltonian H of our interest in the present paper, both properties (2) and (3) fail to hold. Therefore we do not have uniform control of the filtration of the Floer complex (or of the action bounds of the associated Hamiltonian chords). This is a new phenomenon for the localization in the current topological Hamiltonian context. Because of this lack of control of the filtration, we will instead use the more geometric version of thick-thin decomposition mainly using the C^0 property of ϕ_H^1 exploiting the *maximum principle*.

For this purpose, we fix a time-independent almost complex structure J_0 that satisfies $J_0 \equiv J_g$ on V where J_g is the canonical (Sasakian) almost complex structure on V as a subset T^*L which is induced by a Riemannian metric g on L , and suitably interpolated to outsider of U . (We refer to [Fl4], pp 321-323 [Oh2] for the precise description of J_g and J_0 respectively.) We may assume V has J_0 -convex boundary.

Then we will study the equation

$$\begin{cases} \frac{\partial v}{\partial \tau} + J_0 \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H^1(L), v(\tau, 1) \in L \end{cases} \quad (1.8)$$

for each given $J_0 \in \mathcal{J}_\omega$, a time-independent family. We will fix a generic J_0 in the rest of the paper and assume L is transversal to $\phi_H^1(L)$ by considering a C^∞ -small perturbation of H if necessary.

The following thick-thin decomposition of the Floer moduli spaces of (1.8) is a crucial ingredient. This is a variation of Proposition 4.1 [Oh2].

Theorem 1.1 (Compare with Proposition 4.1 [Oh2]). *Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold and let $V \subset \bar{V} \subset U$ be a pair of Darboux neighborhoods of L . Consider a V -engulfable Hamiltonian path ϕ_H . Then whenever $\bar{d}(\phi_H^1, id) \leq \delta$ for any $\delta < d(V, \Theta)$, any solution of v of (1.8) satisfies one of the following alternatives:*

(1) *Either*

$$\text{Image } v \subset D_\delta(L) \subset V \quad (1.9)$$

where $D_\delta(L)$ is the δ -neighborhood of L .

(2) *or $\text{Image } v \not\subset V$. In this case, we also have $\int v^* \omega \geq C(J_0, V)$ where $C(J_0, V) > 0$ is a constant depending only on δ and V .*

We call v a *thin* trajectory if $\text{Image } v \subset V$ and a *thick* trajectory otherwise. We call a thin trajectory *very thin* if it satisfies (1.9) in addition. This theorem basically says that all thin trajectories are indeed very thin and all thick trajectories have area bounded below away from zero. The proof of this theorem is an easy application of maximum principle on the J_0 -convex domain V and the monotonicity formula for the J_0 -holomorphic curves.

This theorem enables us to define the local Floer homology in a well-defined way by counting thin trajectories. We denote this local Floer homology by

$$HF_*^{[id]}(\phi_H^1(L), L; U), \quad \text{or} \quad HF_*^{[id]}(H, (L, L); U).$$

By definition, $HF_*^{[id]}(\phi_H^1(L), L; U)$ is always well-defined *without* any unobstructedness assumption of $L \subset M$ such as exactness or monotonicity of the pair (L, M) or the unobstructedness in the sense of [FOOO1].

Next we consider a 1-parameter family of Hamiltonians (or a 2-parameter family of functions on M) $\mathcal{H} = \{H(s)\}_{0 \leq s \leq 1}$ with $H(0) \equiv 0$ and

$$\max_{s \in [0, 1]} \bar{d}(\phi_{H(s)}^1, id) < \delta \quad (1.10)$$

for a sufficiently small $\delta = \delta_0(M, \omega; J_0)$. We fix an elongation function $\rho : \mathbb{R} \rightarrow [0, 1]$ satisfying

$$\begin{aligned} \rho(\tau) &= \begin{cases} 0 & \tau \leq 0 \\ 1 & \tau \geq 1 \end{cases} \\ \rho' &\geq 0 \end{aligned} \quad (1.11)$$

and define its dual $\tilde{\rho} := 1 - \rho$. Then we consider the Cauchy-Riemann equation with moving boundary condition

$$\begin{cases} \frac{\partial v}{\partial \tau} + J_0 \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_{H(\rho(\tau))}^1(L), v(\tau, 1) \in L, \end{cases} \quad (1.12)$$

and prove the following analog to Theorem 2.1 for chain maps. This is the analogue of the *handle sliding lemma* from [Oh6, Oh10] in which was studied the case with C^2 -smallness of Hamiltonians replaced by the smallness in hamiltonian topology (and also in the Lagrangian context).

Theorem 1.2 (Handle sliding lemma). *Consider the path $\mathcal{H} : s \mapsto H(s)$ of engulfable Hamiltonians $H(s)$ satisfying (1.10) and fix an elongation function $\rho : \mathbb{R} \rightarrow [0, 1]$. Let $J = \{J(s, t)\}_{(s, t) \in [0, 1]^2}$ be the 2-parameter family*

$$J(s, t) = (\phi_{H(s)}^t \circ (\phi_{H(s)}^1)^{-1})_* J_0.$$

Then whenever $\bar{d}(\phi_{H(s)}^1, id) \leq \delta < d(V, \Theta)$, any finite energy solution v of (1.12) satisfies one of the following alternatives:

(1) *Either*

$$\text{Image } v \subset D_\delta(L) \subset V, \quad (1.13)$$

(2) *or $\text{Image } v \not\subset V$. In this case, we also have $\int v^* \omega \geq C(J_0, V)$ where $C(J_0, V) > 0$ is a constant depending only on δ and V .*

Once these thick-thin decomposition results of the Floer moduli spaces for the boundary and for the chain maps are established, essentially the same isolatedness argument as in [Oh2] gives rise to the following computation

Theorem 1.3. *Let $L \subset M$ be as above and U be a Darboux neighborhood of L and $\mathcal{H} : s \mapsto H(s)$ a family of U -engulfable Hamiltonians with $H(0) = 0$. Then if $\max_{s \in [0, 1]} \bar{d}(\phi_{H(s)}^1, id) < \delta$ and $|J_t - J_0|_{C^1} < \delta$ for some time independent J_0 and if J is $(L, \phi_H^1(L))$ -regular, then*

$$HF_*(H, L; J; U) \cong H_*(L; \mathbb{Z}).$$

Remark 1.4. We would like to emphasize that the presence of the engulfable homotopy \mathcal{H} is crucial in the statement of this theorem, because the commonly used the linear homotopy $s \mapsto (1 - s)H_0 + sH_1$ may not be U -engulfable and so may not induce even chain map between the local Floer complex, even when H_0, H_1 are U -engulfable.

Using this local Floer homology, we can assign local spectral invariant which we denote by $\rho_U^{lag}(H; a)$ for $a \in H^*(L; \mathbb{Z})$. We will restrict to the case $PD[M] = 1$. To highlight the localness of the invariant we denote $\rho_U^{lag}(H; 1_0)$ the corresponding invariant. Denote the global spectral invariant associated to 1 by $\rho^{lag}(H; 1)$.

By specializing to the case of zero section o_N of T^*N , it follows from this corollary that, we can define the local Floer complex

$$(CF_*(F; U, T^*N), \partial_U)$$

for any $F \in \mathcal{H}_\delta^{engulf}(T^*N)$ provided $\delta > 0$ is sufficiently small.

When Theorem 2.1 and Theorem 1.2 are applied to the cotangent bundle T^*L , we obtain the following

Corollary 1.4. *Consider a pair of open neighborhoods $V \subset \bar{V} \subset U$ of o_L in T^*L be given where V is J_0 -convex. Assume $\mathcal{H} = \{H(s)\}$ is an engulfable isotopy with $F = H(1)$ satisfying (1.10). Let $J = \{J(s, t)\}_{(s, t) \in [0, 1]^2}$ be the 2-parameter family*

$$J(s, t) = (\phi_{H(s)}^t \circ (\phi_{H(s)}^1)^{-1})_* J_0.$$

Fix an elongation function $\rho : \mathbb{R} \rightarrow [0, 1]$ and consider the equation (1.12). Then whenever

$$\max_{s \in [0, 1]} \bar{d}(\phi_{H(s)}^1, id) < d(V, \Theta),$$

the followings hold:

- (1) *For $F = H(1)$, any solution of v of (1.8) is thin (and so very thin).*
- (2) *Let $J = \{J(s, t)\}_{(s, t) \in [0, 1]^2}$ be the 2-parameter family*

$$J(s, t) = (\phi_{H(s)}^t \circ (\phi_{H(s)}^1)^{-1})_* J_0.$$

Fix an elongation function $\rho : \mathbb{R} \rightarrow [0, 1]$ and consider the equation (1.12). Then if $\max_{s \in [0, 1]} \bar{d}(\phi_{H(s)}^1, id) < \delta$, any finite energy solution v is thin (and so very thin).

When we specialize our construction, using Corollary 1.4, to a J_0 -convex neighborhood of the zero section in the cotangent bundle, we have the following coincidence result.

Theorem 1.5. *Fix an open neighborhood $V \subset T^*L$ of $o_L \subset T^*L$ that is J_0 -convex. Let $\mathcal{H} = \{H(s)\}$ be an engulfable isotopy with $H(0) = 0$ and $H(1) = F$. Then for any $F \in \mathcal{H}_\delta^{engulf}(M; V)$,*

$$\rho_V^{lag}(F; 1_0) = \rho^{lag}(F; 1).$$

One important consequence of this theorem is the inequality

$$\rho_V^{lag}(F; 1_0) \leq E^-(F). \quad (1.14)$$

We apply the above constructions to the graph

$$\text{Graph } \phi_F^1 = \{(\phi_H^1(x), x) \mid x \in M\} \subset M \times M$$

of engulfable Hamiltonian F on M satisfying

$$\bar{d}(\phi_F^1, id) < \delta \quad (1.15)$$

for a sufficiently small $\delta > 0$. We define

$$\mathcal{H}_\delta^{engulf}(M) \subset C^\infty([0, 1] \times M, \mathbb{R})$$

to be the set of such Hamiltonian F 's, and call the associated Hamiltonian path an engulfable Hamiltonian C^0 -approximate loop.

Consider $\mathcal{U} \subset \mathcal{L}_0(M)$ defined by

$$\mathcal{U} = \mathcal{U}(U_\Delta) = \{\gamma \in \mathcal{L}_0(M) \mid (\gamma(t), \gamma(0)) \in U_\Delta\}.$$

We define the local Floer homology

$$HF_*^{[id]}(F, J; \mathcal{U}), \quad \mathcal{U} \subset \mathcal{L}_0(M)$$

by counting the ‘thin’ trajectories such that their images are contained in a neighborhood \mathcal{U} of the set of constant paths in M . Again we would like to emphasize that we have not control of the area or filtrations unlike the C^2 -small case. In this

regard, it would be really more appropriate to call them *thin* trajectories in the current localization of using the maximum principle, rather than ‘short’ trajectories adopted by Chekanov [Che1, Che2].

We denote by $\rho_{\mathcal{U}}^{\text{ham}}(\phi_F; 1) = \rho_{\mathcal{U}}^{\text{ham}}(\underline{F}; 1)$ the (local) spectral invariant associated to $1 \in H^*(L)$.

From now on, we will always assume that all the Hamiltonians in the rest of the paper are engulfable one way or the other, unless otherwise said explicitly.

We would like to thank D. McDuff and H. Hofer for pointing out a crucial gap in our $W^{1,p}$ -precompactness proof in the previous version of the present paper. This forces us to abandon the area argument to obtain the thick-thin decomposition of Floer moduli spaces but exploit the maximum principle instead to obtain a decomposition result that will do our purpose of extracting the local Floer complex out of the global Floer complex.

2. LOCAL FLOER COMPLEX FOR C^2 -SMALL H ; REVIEW

In this section, we recall the construction of the local version of the Floer homology $HF_*(H; L, L)$ from [Oh2] which singles out the contribution from the Floer trajectories whose images are contained in a given Darboux neighborhood U of L in M , provided

$$\phi_{H^t}(L) \subset V \subset \overline{V} \subset U \quad (2.1)$$

for all $t \in [0, 1]$. It also holds that $HF(H; L, L) \cong HF(F; L, L)$ provided there exists a family $\mathcal{H} = \{H(s)\}_{s \in [0, 1]}$ such that

- (1) $s \mapsto H(s)$ is a smooth path (and so $(s, t) \mapsto \phi_{H(s)}^t$ is a smooth two-parameter family) and $H(0) = H$, $H(1) = F$
- (2) The inclusions (2.1) hold for all $s \in [0, 1]$ for a family of neighborhoods $V(s) \subset \overline{V}(s) \subset U$.

This construction was introduced by Floer in [Fl3] in the Hamiltonian context which was further amplified in [Oh2] in the context of Lagrangian Floer homology. It is also proved in [Oh2] that this local contribution depends only on the pair (L, U) and so we can carry out its computation for the pair (o_L, V) where $V \subset T^*L$ is a neighborhood of the zero section $o_L \cong L$. We refer to [Oh2] for the full details of construction thereof.

We denote this local Floer homology by

$$HF(\phi_H^1(L), L; U).$$

Furthermore by definition, $HF(\phi_H^1(L), L; U)$ is always well-defined *without* any unobstructedness assumption of $L \subset M$ like exactness of ω or monotonicity of the pair (L, M) .

The construction is based on the following flexible notion that Floer introduced in [Fl3] in the Hamiltonian setting. This is in turn the Floer theoretic analog of the notion of *isolating block* introduced by Conley [Co] in dynamical system. The definition can be formulated in a more general abstract setting but we will focus on the current geometric context.

Let $U \subset M$ be a Darboux neighborhood of L .

Definition 2.1. Let $L' \subset U$ be any compact Lagrangian submanifold and $J \in \mathcal{P}(\mathcal{J}_\omega)$. Consider $\mathcal{M}_1(L', L; J)$ the set of Floer trajectories with one marked point

which could be either an interior or a boundary point and its evaluation map $ev : \mathcal{M}_1(L', L; J) \rightarrow M$. Define the subset

$$\widetilde{\mathcal{M}}(L', L; J; U) := \{u \in \widetilde{\mathcal{M}}(L', L; J) \mid \text{Image } u \subset U\}$$

and its evaluation image

$$\mathcal{S}(L', L; J; U) = ev(\mathcal{M}_1(L', L; J; U)) \subset U$$

We call $\mathcal{S}(L', L; J; U)$ an *invariant set* in U and say that $\mathcal{S}(L', L; J; U)$ is *isolated* in U (under the Cauchy-Riemann flow) if $\overline{\mathcal{S}}(L', L; J; U) \subset U$ where $\overline{\mathcal{S}}(L', L; J; U)$ is the closure of $\mathcal{S}(L', L; J; U)$ in M . When $\mathcal{S}(L', L; J; U)$ is isolated, we call $\overline{\mathcal{S}}(L', L; J; U)$ the *maximal invariant set* in U .

In this regard the following result on the thick-thin dichotomy of Floer trajectories plays a crucial role in [Oh2]. For each given tame almost complex structure J_0 on (M, ω) , we define the constants

$$\begin{aligned} \sigma_S(M, J_0) &= \inf\{\omega([u]) \mid u : S^2 \rightarrow M \text{ non-constant and} \\ &\quad \text{satisfying } \overline{\partial}_{J_0} u = 0\}. \end{aligned}$$

When a compact Lagrangian submanifold $L \subset (M, \omega)$ is given, we also consider

$$\sigma_D(M, L, J_0) := \inf_v \left\{ \int v^* \omega, \mid v \text{ non-constant solution of } \overline{\partial}_{J_0} v = 0 \right\}.$$

Then we set

$$\sigma(M, L, J_0) = \min\{\sigma_S(M, J_0), \sigma_D(M, L, J_0)\}, \quad \sigma(M, L) = \sup_{J_0 \in \mathcal{J}_\omega} \sigma(M, L, J_0).$$

In the rest of the present paper, we will denote $\sigma_S(M, J_0)$, $\sigma_D(M, L, J_0)$ and others by $A_S(\omega, J_0)$, $A_D(\omega, L, J_0)$ and

$$A(\omega, L) = \sigma(M, L) = \sup_{J_0 \in \mathcal{J}_\omega} \sigma(M, L, J_0). \quad (2.2)$$

following the notations of [Oh2] and other papers of the present author such as [Oh8].

Theorem 2.1 (Proposition 4.1 [Oh2]). *Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold and let $V \subset \overline{V} \subset U$ be a pair of Darboux neighborhoods of L . Let $0 < \varepsilon < \frac{1}{2}A(\omega, L; J_0)$ be any given constant. Then there exists $\delta = \delta(\varepsilon) > 0$ depending only on ε (and (M, ω)) such that whenever $\|H\|_{C^2} \leq \delta$, and $\|J - J_0\|_{C^1} \leq \delta$, any solution of v of (3.5) with*

$$\int v^* \omega \leq A(\omega, L; J_0) - \varepsilon$$

automatically satisfies $\int v^ \omega \leq \varepsilon$. Furthermore*

$$\text{Image } v \subset V \subset \overline{V} \subset U.$$

Chekanov [Che2] paraphrases the property spelled out in this theorem by saying *every short trajectory is indeed very short*.

Now we define the notion of *continuation* of maximal invariant sets.

Definition 2.2. Consider

$$(J^{para}, H^{para}) \in \text{Map}([0, 1]^2, \mathcal{J}_\omega) \times C^\infty([0, 1]^2 \times M, \mathbb{R})$$

and an open subset $U^{para} \subset [0, 1] \times M$. We call $(J^{para}, H^{para}, U^{para})$ a *continuation* between the maximal invariant sets $\mathcal{S}_0 \subset U^0$ and $\mathcal{S}_1 \subset U^1$ if it satisfies the following:

(1) For each $s \in [0, 1]$ and all $t \in [0, 1]$,

$$L^s \subset U^s := \{x \in M \mid (x, s) \in U^{para}\}.$$

(2)

$$\mathcal{S}_s := \mathcal{S}(J^s, (L^s, L); U^s)$$

is isolated in U^s for all $s \in [0, 1]$.

We apply this definition to a family of Lagrangian submanifolds which are Hausdorff-close to L in the following sense.

Definition 2.3. We call a Lagrangian submanifold $L' \subset (M, \omega)$ *exact relative to L* if there is a Darboux neighborhood $U \supset L$ such that $L' \subset U$ and is exact in $U \cong V \subset T^*L$.

Once we have set up these definitions, the following is easy to prove.

Lemma 2.2. *Let $L \subset M$. Let L' be exact relative to L and intersect L transversally. Suppose that $\mathcal{S}(L', L; J; U)$ is isolated in U . Then there exists a C^∞ perturbation J' of J for which $\mathcal{M}(L', L; J'; U)$ is Fredholm regular and $\mathcal{S}(L', L; J'; U)$ remains isolated in U . In particular, for any pair $x, y \in L \cap L'$ with $\mu(x; U) - \mu(y; U) = 1$ $\mathcal{M}(x, y; J'; U) \subset \mathcal{M}(L', L; J'; U)$ has finite cardinality.*

Now suppose $(L', L; J; U)$ and J' are as in Lemma 2.2. We define $n_U(x, y; J')$ by

$$n_U(x, y; J') := \# \text{ of isolated trajectories in } \mathcal{M}((x, y); J; U).$$

If $L' = \phi_H^1(L)$, then we can define an integer $n_U(x, y; J')$ using the coherent orientation established in [Oh3]. We refer to section 17.2 [Oh13] for the details of the proof.

Theorem 2.3. *Suppose $(L', L; J; U)$ is as in Lemma 2.2. Then for any small perturbation J' of J for which $\mathcal{M}(L', L; J'; U)$ is Fredholm regular, Then the homomorphism*

$$\partial_U : CF(L', L; J'; U) \rightarrow CF(L', L; J'; U), \quad \partial_U x = \sum_{y \in L \cap \phi_H^1(L)} \langle \partial_U x, y \rangle y$$

satisfies $\partial_U \circ \partial_U = 0$. And the corresponding quotients

$$HF^*(L', L; J'; U) = \ker \partial_U / \text{im } \partial_U$$

are isomorphic under the continuation $(\mathcal{S}^{para}, J^{para}, H^{para}, U^{para})$ as long as the continuation is Floer-regular at the ends $s = 0, 1$.

This completes the following computation by the arguments given in [Fl4], [Oh2]. We refer to section 3 and 4 [Oh2] for the full details of the proof.

Finally the following theorem is proved by deforming H to the C^2 -small time-independent Morse function f and applying Floer's argument given in [Fl4] (See [Fl4, Oh2] for the details.)

Theorem 2.4. *Let $L \subset M$ be as above and U be a Darboux neighborhood of L . Then if $\|H\|_{C^2} \leq \varepsilon_3$ and $|J_t - J_0|_{C^1} < \varepsilon_3$ for some time independent J_0 and if J is $(L, \phi_H^1(L))$ -regular, then*

$$HF((L, L), H; U) \cong HF(\phi_H^1(L), L; J; U) \cong H_*(L; \mathbb{Z}).$$

One important point of this proof in hindsight, which is not conspicuous but implicit in the argument used in [Oh2, Che2], is that if $\|H\|_{C^2}$ is sufficiently small the chain map Floer moduli space associated to the linear homotopy

$$s \mapsto (1-s)f + sH$$

is isolated in V . To obtain the required isolatedness, one needs to assume that the Hamiltonian is C^2 -small. Then one has to partition the homotopy into smaller segments to obtain this C^2 -smallness and then apply the *adiabatic chain homotopy* as in [Oh6].

This isolating property of the linear homotopy will fail for the Hamiltonian H of our main interest in the present paper and so we cannot use this linear homotopy to establish this isomorphism statement. (See Remark 8.2.) This is one of the reasons why the presence of engulfable isotopy

$$H(s) \in \mathcal{H}_\delta^{engulf}(L; V)$$

satisfying $H(0) = 0$ and $H(1) = H$ and its isolating property (which we call *Handle sliding lemma*, Theorem 6.1, following the terminology used in [Oh6]) enters in the proof of a similar isomorphism theorem stated in Theorem 2.4.

This ends the discussion of the local Floer homology of L for the case when H is C^2 -small and J is C^1 -close to J_0 . For the rest of the paper, we will extend this construction of local Floer homology to the case of those Hamiltonians H that is engulfable and its time-one map is sufficiently C^0 -small.

3. COMPARISON OF TWO CAUCHY-RIEMANN EQUATIONS

For each given pair (J, H) , we consider the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\ u(\tau, 0) u(\tau, L_1) \in L \end{cases} \quad (3.1)$$

which defines the Floer complex $CF_*(L, L; H)$ generated by the set $Chord(H; L, L)$ defined by

$$Chord(H; L, L) := \{z : [0, 1] \rightarrow M \mid \dot{z} = X_H(t, z), z(0), z(1) \in L\}. \quad (3.2)$$

We call any such element z in $Chord(H; L, L)$ a Hamiltonian chord of L . This Cauchy-Riemann equation is called the *dynamical version* in [Oh3].

On the other hand, one can also consider the *genuine* Cauchy-Riemann equation

$$\begin{cases} \frac{\partial v}{\partial \tau} + J^H \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H^1(L), v(\tau, 1) \in L \end{cases} \quad (3.3)$$

for the path $u : \mathbb{R} \rightarrow \mathcal{P}(\phi_H^1(L), L)$

$$\mathcal{P}(\phi_H^1(L), L) = \{\gamma : [0, 1] \rightarrow T^*N \mid \gamma(0) \in \phi_H^1(L), \gamma(1) \in L\}$$

and $J_t^H = (\phi_H^t(\phi_H^1)^{-1})_* J_t$. We call this version the *geometric version*.

We now describe the geometric version of the Floer homology in some more details. The upshot is that there is a filtration preserving isomorphisms between the dynamical version and the geometric version of the Lagrangian Floer theories.

We denote by $\widetilde{\mathcal{M}}(L_H, L; J^H)$ the set of finite energy solutions and $\mathcal{M}(L_H, L; J^H)$ to be its quotient by \mathbb{R} -translations. In the unobstructed case [FOOO1], this gives rise to the geometric version of the Floer homology $HF_*(\phi_H^1(L), L, \widetilde{J})$ of the type

[Fl1] whose generators are the intersection points of $\phi_H^1(L) \cap L$. An advantage of this version is that it depends only on the Lagrangian submanifold $(\phi_H^1(L), L)$ but only loosely on H .

The following is a straightforward to check but is a crucial lemma.

Lemma 3.1. (1) *The map $\Phi_H : \phi_H^1(L) \cap L \rightarrow \text{Chord}(H; L, L)$ defined by*

$$x \mapsto z_x^H(t) = \phi_H^t(\phi_H^{-1}(x))$$

gives rise to the one-one correspondence between the set $\phi_H^1(L) \cap L \subset \mathcal{P}(\phi_H^1(L), L)$ as constant paths and the set of solutions of Hamilton's equation of H .

(2) *The map $a \mapsto \Phi_H(a)$ also defines a one-one correspondence from the set of solutions of (3.6) and that of*

$$\begin{cases} \frac{\partial v}{\partial \tau} + J^H \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H^1(L), v(\tau, 1) \in L \end{cases} \quad (3.4)$$

where $J^H = \{J_t^H\}$, $J_t^H := (\phi_H^t(\phi_H^1)^{-1})^ J_t$. Furthermore, (3.4) is regular if and only if (3.1) is regular.*

Once we have transformed (3.1) to (3.4), we can further deform J^H to the constant family J_0 and consider

$$\begin{cases} \frac{\partial v}{\partial \tau} + J_0 \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H^1(L), v(\tau, 1) \in L. \end{cases} \quad (3.5)$$

This latter deformation preserves the filtration of the associated Floer complexes [Oh3]. A big advantage of considering this equation is that it enables us to study the behavior of spectral invariants for a sequence of L_i converging to L in Hausdorff distance.

We also study the above comparison for the moving boundary condition for a family $\mathcal{H} = \{H(s)\}_{s \in [0,1]}$. For such a family, we consider the geometric version first

$$\begin{cases} \frac{\partial v}{\partial \tau} + J_0 \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_{H(\rho(\tau))}^1(L), v(\tau, 1) \in L \end{cases} \quad (3.6)$$

for the path $v : \mathbb{R} \times [0, 1] \rightarrow M$. If we define a map $u : \mathbb{R} \times [0, 1] \rightarrow M$

$$u(\tau, t) = \phi_{H(\rho(\tau))}^t(\phi_{H(\rho(\tau))}^1)^{-1}(v(\tau, t)),$$

a simple calculation proves that u satisfies $u(\tau, 0), u(\tau, 1) \in L$ and

$$\begin{cases} \frac{\partial u}{\partial \tau} - X_{K(\rho(\tau))}(u) + J \left(\frac{\partial u}{\partial t} - X_{H(\rho(\tau))}(u) \right) = 0 \\ u(\tau, 0), u(\tau, 1) \in L \end{cases} \quad (3.7)$$

where K is the s -Hamiltonian generating the Hamiltonian vector field

$$X_K(s, t, x) := \frac{\partial \phi}{\partial s}(\phi^{-1}(s, t, x))$$

of the 2-parameter family $(s, t) \mapsto \phi(s, t) = \phi_{H(s)}^t \phi_{H(s)}^{-1}$ and $J = J(s, t) = (\phi(s, t))_* J_0$. We would like to highlight the presence of the terms $X_{K(\rho(\tau))}(u)$ in the above equation for u and the definition of energy of u . The associated *off-shell* energy of (3.7)

is given by

$$E_{(H,K),J;\rho}(u) = \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial u}{\partial \tau} - X_{K(\rho(\tau))}(u) \right|_J + \left| \frac{\partial u}{\partial t} - X_{H(\rho(\tau))}(u) \right|_J^2 dt d\tau. \quad (3.8)$$

which coincides with

$$\int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial u}{\partial t} - X_{H(\rho(\tau))}(u) \right|_J^2 dt d\tau$$

on shell. The proof of the on-shell identities

$$\int v^* \omega = E_{J_0}(v) = E_{(H,K),J;\rho}(u)$$

is straightforward and so omitted. With these correspondences, we have the obvious analog to Lemma 3.1 for the moving boundary condition, whose precise statement we omit.

4. THICK-THIN DECOMPOSITION FOR ENGULFABLE C^0 -APPROXIMATE LOOP ϕ_H

This section is a modification of section 3 of [Oh2] which treats the case of C^2 -small perturbation of Hamiltonians H . In this section, we will replace the condition of ϕ_H being C^1 -small by that of ϕ_H being C^0 -small.

Consider a sequence $v : \mathbb{R} \times [0, 1] \rightarrow M$ of solutions of (3.5) associated to H and J_0 . We re-state Theorem 2.1 here.

Theorem 4.1. *Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold and let $V \subset \overline{V} \subset U$ be a pair of Darboux neighborhoods of L . Consider a V -engulfable Hamiltonian path ϕ_H . Then there exists $\delta > 0$ depending only on ε (and (M, ω)) such that whenever $\overline{d}(\phi_H^1, id) \leq \delta$, any solution of v of (3.5) satisfies one of the following alternatives:*

- (1) Image $v \subset V$ and $\max d(v(z), o_L) \leq d_H(\phi_H^1(L), L)$,
- (2) Image $v \not\subset V$ and $\int v^* \omega \geq C(J_0, V)$ where $C(J_0, V) > 0$ is a constant depending only on V .

Proof. Suppose that Image $v \not\subset V$. Then there exists a point $v(z) \notin V$ and so

$$d(v(z), v(\partial(\mathbb{R} \times [0, 1]))) \geq \min\{d_H(\partial V, \phi^1(L)), d_H(\partial V, o_L)\}.$$

Then monotonicity formula implies

$$\int v^* \omega \geq C' \cdot (\min\{d_H(\partial V, \phi^1(L)), d_H(\partial V, o_L)\})^2$$

where C' is the monotonicity constant of (M, ω, J_0) in the monotonicity formula. Considering $\delta < \frac{1}{4} \cdot d_H(\partial V, o_L)$ and setting

$$C(J_0, V) := \frac{1}{2} C' \cdot (d_H(\partial V, o_L) - \delta)^2 \geq \frac{1}{4} C' d_H(\partial V, o_L)^2$$

(2) follows.

For the curve v of the type (1), the maximum principle applied to J_0 -holomorphic curves contained in the Darboux neighborhood, we obtain the maximum distance of $v(z)$ from L is achieved on the boundary $\mathbb{R} \times \{0, 1\}$. But by the boundary condition, we have

$$\max_{z \in \mathbb{R} \times \{0, 1\}} d(v(z), o_L) \leq d_H(\phi_H^1(L), L).$$

This finishes the proof. \square

Remark 4.1. We would like to note that the area property for the trajectories v of the type (2) spelled out as

$$\int v^* \omega \geq C(J_0, V)$$

with constant $C(J_0, V) > 0$ independent of H will not be used in this paper. However this property will lead to the so called gapped property in the sense of [FOOO1] for the associated decomposition $\partial = \partial_{(0)} + \partial'$ to be obtained in the next section. We believe that this gappedness will be important when one wants to generalize the homotopy invariance proved in [Oh15] to the *irrational* (M, ω) . We will come back to this elsewhere.

5. LOCAL FLOER COMPLEX OF ENGULFABLE C^0 -APPROXIMATE LOOP ϕ_H

In this section, we will make precise the meaning of local Floer complex for the Hamiltonian H for which $\bar{d}(\phi_H^1, id)$ is small. From now on, we will fix a pair of Darboux neighborhood $V \subset \bar{V} \subset U$ of L in M and assume H is V -engulfable, i.e., satisfies

$$\phi_{H^t}(L) \subset V \subset \bar{V} \subset U \quad (5.1)$$

for all $t \in [0, 1]$.

Next we recall the Lagrangian analogue of the Novikov ring $\Gamma_\omega = \Gamma(M, \omega)$ from [FOOO1]. Denote by $I_\omega : \pi_2(M, L) \rightarrow \mathbb{R}$ the evaluations of symplectic area. We also define another integer-valued homomorphism $I_\mu : \pi_2(M, L) \rightarrow \mathbb{Z}$ by

$$I_\mu(\beta) = \mu(w^*TM, (\partial w)^*TL)$$

which is the Maslov index of the bundle pair $(w^*TM, (\partial w)^*TL)$ for a (and so any) representative $w : (D^2, \partial D^2) \rightarrow (M, L)$ of β .

Definition 5.1. We define

$$\Gamma_{(\omega, L)} = \frac{\pi_2(M, L)}{\ker I_\omega \cap \ker I_\mu}.$$

and $\Lambda(\omega, L)$ to be the associated Novikov ring.

We briefly recall the basic properties on the Novikov ring $\Lambda_{(\omega, L)}(R)$ where R is a commutative ring where R could be \mathbb{Z}_2 , \mathbb{Z} or \mathbb{Q} for example. We will just use the letter R for the coefficient ring which we do not specify. Basically R will be \mathbb{Q} when the associated moduli space is orientable as in the case of $\text{Graph } \phi_H^1$ for a Hamiltonian diffeomorphism ϕ_H^1 which is of our main interest.

We put

$$q^\beta = T^{\omega(\beta)} e^{\mu_L(\beta)},$$

and

$$\deg(q^\beta) = \mu_L(\beta), \quad E(q^\beta) = \omega(\beta)$$

which makes $\Lambda_{(\omega, L)}$ and $\Lambda_{0, (\omega, L)}$ become a graded ring in general. We have the canonical valuation $\nu : \Lambda_{(\omega, L)} \rightarrow \mathbb{R}$ defined by

$$\nu \left(\sum_{\beta} a_{\beta} T^{\omega(\beta)} e^{\mu_L(\beta)} \right) = \min \{ \omega(\beta) \mid a_{\beta} \neq 0 \}$$

It induces a valuation on $\Lambda_{(\omega, L)}$ which induces a natural filtration on it. This makes $\Lambda_{(\omega, L)}$ a filtered graded ring. For a general Lagrangian submanifold, this ring may not even be Noetherian but it is so if L is rational, i.e., $\Gamma(L; \omega)$ is discrete.

Now consider a nondegenerate Hamiltonian H among those given in Theorem 4.1. Following [Che2] we say that two elements of $\text{Crit } \mathcal{A}_H$ are equivalent if they belong to the same connected component of the set

$$\pi^{-1}(\{\gamma \in \Omega(L, L) \mid \gamma([0, 1]) \subset U\}) \subset \widetilde{\Omega}(L, L).$$

Then the projection $\pi : \widetilde{\Omega}(L, L) \rightarrow \Omega(L, L)$ bijectively maps each equivalence class of $\text{Crit } \mathcal{A}_H$ to $\text{Chord}(H; L, L)$. There is a ‘canonical equivalence class’ represented by the pairs

$$[z, w_z]$$

where $z \in \text{Chord}(H; L, L)$ and w_z is the (homotopically) unique cone-contraction of z to a point in L .

We denote this equivalence class by $\text{Crit}^{[id]} \mathcal{A}_H \subset \text{Crit } \mathcal{A}_H$. This induces the natural $\Gamma_{(\omega, L)}$ -action on $\text{Crit } \mathcal{A}_H$ which gives rise to the bijection

$$\text{Crit}^{[id]} \mathcal{A}_H \times \Gamma_{(\omega, L)} \rightarrow \text{Crit } \mathcal{A}_H.$$

Remark 5.2. Note that for any $[z, w] \in \text{Crit}^{[id]} \mathcal{A}_H$, (5.1) implies $z(t) \in V$ since $z(0) \in \phi_H^1(L)$. Therefore the action value $\mathcal{A}_H([z, w])$ will not change even if we cut-off H outside V .

We denote

$$\text{Crit}^{[g]} \mathcal{A}_H = g \cdot \text{Crit}^{[id]} \mathcal{A}_H, \quad g \in \Gamma_{(\omega, L)}.$$

With this notation, we have $\text{Crit}^{[id]} \mathcal{A}_H = \text{Crit}^{[id]} \mathcal{A}_H$. Then we denote their associated R -module by

$$CF_*^{[g]}((L, L), H; U), \quad CF_*^{[id]}((L, L), H; U) = CF_*^{[id]}((L, L), H; U).$$

We want to remark that $CF_*^{[id]}((L, L), H; U)$ is the one that was used in [Oh2] for the case of C^2 -small cases.

The above discussion in turn gives rise to the isomorphism

$$CF_*^{[g]}((L, L), H; U) \otimes_R \Lambda_{(\omega, L)} \cong CF_*((L, L); H)$$

as $\Lambda_{(\omega, L)}$ -module for each $g \in \Lambda_{(\omega, L)}$. Following [Che1, Che2], we denote

$$\text{leng}(u) := E_J(u) = E_{J_0}(v) = \text{area}(v). \quad (5.2)$$

Now we note that the Floer (pre)-boundary map

$$\partial : CF_*((L, L); H) \rightarrow CF_*((L, L); H)$$

is $\Lambda_{(\omega, L)}$ -equivariant and has the decomposition

$$\partial = \sum_{\lambda \in \mathbb{R}_{\geq 0}} \partial_\lambda$$

where ∂_λ is the contribution arising from $u \in \mathcal{M}(L, L; H)$ with

$$\text{leng}(u) = \lambda > 0.$$

We now define

$$\partial = \partial_{(0)} + \partial' \quad (5.3)$$

where $\partial_{(0)}$ is the sum of contribution of thin trajectories and ∂' that of thick trajectories.

We also denote $u \in \text{supp } \partial$, $\text{supp } \partial_{(0)}$, and $\text{supp } \partial'$ respectively, if the map u nontrivially contributes to the corresponding operators.

Definition 5.3. We call $(CF_*^{[id]}((L, L), H; U), \partial_U)$ the *local Floer complex* of H in U which is defined to be

$$\begin{aligned} CF_*^{[id]}((L, L), H; U) &= R \cdot \{\text{Crit}^{[id]} \mathcal{A}_H\}, \\ \partial_U &= \partial_{(0)} \Big|_{CF_*^{[id]}((L, L), H; U)}. \end{aligned}$$

The $\Lambda_{(\omega, L)}$ -equivariance of ∂ gives rise to

$$\widehat{g} \circ \partial_{(0)} \Big|_{CF_*^{[id]}((L, L), H; U)} = \partial_{(0)} \Big|_{CF_*^{[g]}((L, L), H; U)} \circ \widehat{g}$$

and \widehat{g} carries a natural weight given by

$$\mathcal{A}_F(g \cdot [z, w]) - \mathcal{A}_F([z, w]), [z, w] \in \text{Crit } \mathcal{A}_F$$

which does not depend on the choice of $[z, w] \in \text{Crit } \mathcal{A}_F$. In fact this real weight is nothing but the value $\omega([g])$.

Proposition 5.1. *Let $\delta > 0$ where δ is the constant given in Theorem 4.1. Then $\partial_U^2 = 0$ and so the local Floer homology*

$$HF_*^{[id]}((L, L), H; U) = \ker \partial_U / \text{im } \partial_U$$

is well-defined.

Proof. Since all the thin trajectories have their image contained in the Darboux neighborhood U , concatenations of thin trajectories also thin and the thin part of Floer moduli spaces for the pair $(\phi_H^1(L), L)$ does not bubble-off. This then immediately finishes the proof. \square

In the next section we will compute the group $HF_*^{[id]}((L, L), F; U)$, when $F = H(1)$ for a 2-parameter family $\mathcal{H} = \{H(s)\}_{s \in [0, 1]}$ with $H(0) = 0$ and $H(s) \in \mathcal{H}_\delta^{\text{engulf}}(M)$. We denote by

$$\overline{d}(\phi_{\mathcal{H}}^1, id) := \max_{s \in [0, 1]} \overline{d}(\phi_{H(s)}^1, id)$$

the C^0 -distance of \mathcal{H} to the constant family id .

6. HANDLE SLIDING LEMMA FOR ENGULFABLE ISOTOPY OF C^0 -APPROXIMATE LOOPS

In this section, we examine another important element in the chain level theory, the *handle sliding lemma* introduced in [Oh6] for the Hamiltonian H that is sufficiently C^2 -small. We will consider the lemma in the Lagrangian setting over the path $s \mapsto H(s)$ for $\mathcal{H} = \{H(s)\}_{s \in [0, 1]} \subset \mathcal{H}_\delta^{\text{engulf}}(M)$ with $H(0) \equiv 0$ for δ sufficiently small. Again the smallness will depend only on (M, ω) .

We recall that the Floer chain map $h_{H^\rho} : CF_*(H^0) \rightarrow CF_*(H^1)$ is defined by considering the non-autonomous equation

$$\begin{cases} \frac{\partial u}{\partial \tau} - X_{K(\rho(\tau))}(u) + J \left(\frac{\partial u}{\partial t} - X_{H(\rho(\tau))}(u) \right) = 0 \\ \lim_{\tau \rightarrow -\infty} u(\tau) = z^-, \lim_{\tau \rightarrow \infty} u(\tau) = z^+ \end{cases} \quad (6.1)$$

or equivalently considering (3.5). Here we re-state Theorem 1.2 here.

Theorem 6.1 (Handle sliding lemma). *Consider the path $\mathcal{H} : s \mapsto H(s)$ of engulfable Hamiltonians $H(s)$ satisfying (1.10) and fix an elongation function $\rho : \mathbb{R} \rightarrow [0, 1]$. Let $J = \{J(s, t)\}_{(s, t) \in [0, 1]^2}$ be the 2-parameter family*

$$J(s, t) = (\phi_{H(s)}^t \circ (\phi_{H(s)}^1)^{-1})_* J_0.$$

Then if $\bar{d}(\phi_{H(s)}^1, id) < d(V, \Theta)$, any finite energy solution v of (1.12) satisfies one of the following alternatives:

- (1) Image $v \subset V$,
- (2) Image $v \not\subset V$. In this case, $\int v^* \omega \geq C(J_0, V)$ where $C(J_0, V) > 0$ is a constant depending only on J_0 and V .

Proof. The proof is the same as that of Theorem 4.1 and so omitted. \square

Now Theorem 6.1 together with this area estimate then enable us to decompose the Floer-Piunikhin (pre)-chain map $\Psi_{\mathcal{H}} : C_*(L) \rightarrow CF_*(H(1))$ into the thick-thin decomposition

$$\Psi_{\mathcal{H}} = \Psi_{\mathcal{H},(0)} + \Psi'_{\mathcal{H}} \quad (6.2)$$

similar to (5.3). Again it follows from Theorem 1.2 that those v 's contributing non-trivially to $\Psi_{\mathcal{H},(0)}$ are very thin and those contributing to $\Psi'_{\mathcal{H}}$ has area bigger than $C(J_0, V)$.

We refer to section 5.3 [FOOO1] or section 5 [FOOO3] for the details of the construction of the Floer-Piunikhin (pre)-chain map $\Psi_{\mathcal{H}}$.

Remark 6.1. The above Handle sliding lemma can be also proved by the same argument for the Floer chain map between f and $H(1)\#f$ when $|f|_{C^2}$ is sufficiently small relative to $C(V, J_0)$. This way one can avoid using the Bott-Morse version of Floer chain map, the Floer-Piunikhin (pre)-chain map $\Psi_{\mathcal{H}}$.

7. COMPUTATION OF LOCAL FLOER HOMOLOGY $HF_*^{[id]}((L, L), H; U)$

The role of the C^2 -smallness in the construction of local Floer complex

$$HF_*^{[id]}((L, L), H; U)$$

in [Oh2] was two-fold. One is to make its flow ϕ_H C^1 -small which gives rise to a thick-thin decomposition of Floer complex. The other is for the construction of (local) chain isomorphism between the singular complex of L and the Floer complex $CF_*^{[id]}((L, L), H; U)$ for which one needs to avoid bubbling (especially disc-bubbling) to ensure the chain isomorphism property of the Floer-Piunikhin's continuation map. For the latter purpose, we need to obtain some estimates of the filtration change for the Floer chain map between the identity path and ϕ_H over the family

$$\mathcal{H} : s \mapsto H(s), \quad s \in [0, 1].$$

In the present context, we do not have much control over the filtration change under the chain map we construct, *even if one uses the adiabatic chain map mentioned before*: Since we do not have any restriction on the C^2 -norm of H , we will not have much control on the mesh of the partitions we make for the given approximating sequence H_i . To overcome this lack of control of the filtration, we use Conley and Floer's idea of continuation of maximal invariant sets [Co, Fl3, Oh2].

For given one parametric family

$$(J^{para}, H^{para}) \in Map([0, 1]^2, \mathcal{J}_\omega) \times C^\infty([0, 1]^2 \times M, \mathbb{R})$$

with $H^{para} = \mathcal{H}$ with $H(0) = 0$, we define a *continuation* U^{para} between the maximal invariant sets $\mathcal{S}_0 \subset U^0$ and $\mathcal{S}_1 \subset U^1$ to be an open subset of $[0, 1] \times M$ that satisfies

- (1) For each $s \in [0, 1]$ and all $t \in [0, 1]$,

$$L^s \subset U^s := \{x \in M \mid (x, s) \in U^{para}\}.$$

- (2)

$$\mathcal{S}_s := \mathcal{S}(J^s, (L^s, L); U^s)$$

is isolated in U^s for all $s \in [0, 1]$.

The proof of isolatedness of $\mathcal{S}(J_0, (\phi_{H(s)}^1(L), L); V)$ for $0 \leq s \leq 1$ will be based on the parametric version of Theorem 4.1.

Once we have set up these definitions and isolatedness, it immediately gives rise to the following theorem

Theorem 7.1. *Suppose $(L', L; J; U)$ is as in Lemma 2.2. Suppose $H \in \mathcal{H}_\delta^{engulf}(L; U)$ for a sufficiently small $\delta = \delta(\varepsilon) > 0$. Then for any small perturbation J' of J for which $\mathcal{M}(L', L; J'; U)$ is Fredholm regular,*

- (1) *the homomorphism*

$$\partial_U : CF(L', L; J'; U) \rightarrow CF(L', L; J'; U), \quad \partial_U x = \sum_{y \in L \cap \phi_H^1(L)} \langle \partial_U x, y \rangle y$$

satisfies $\partial_U \circ \partial_U = 0$.

- (2) *And the corresponding quotients*

$$HF(L, L; (H, J'); U) \cong HF^*(L', L; J'; U) = \ker \partial_U / \text{im } \partial_U$$

are isomorphic under the continuation $(\mathcal{S}^{para}, J^{para}, H^{para}, U^{para})$ as long as the continuation is Floer-regular at the ends $s = 0, 1$.

After we establish this continuation invariance, we can apply it to the family \mathcal{H} with $H(0) = 0$ and prove the following theorem.

Theorem 7.2. *Consider $\mathcal{H} = \{H(s)\} \subset \mathcal{H}_\delta^{engulf}(M)$ with $H(0) = 0$. Then whenever $0 < \delta < d(V, \Theta)$,*

$$HF^{[id]}(\phi_H^1(L), L; J'; U) \cong H_*(L; \mathbb{Z})$$

for any J' sufficiently close to J_0 in C^∞ -topology.

Proof. We consider the homotopy

$$\mathcal{H} : s \mapsto H(s)$$

and its reversal.

Using the isolatedness of thin trajectories in Theorem 4.1 and Theorem 6.1, we define the local Floer-Piunikhin (pre)-chain maps

$$\begin{aligned} \Psi_{\mathcal{H}^\rho, (0)} : CF_*^{[id]}((L, L), 0; U) &\rightarrow CF_*^{[id]}((L, L), H; U), \\ \Phi_{\mathcal{H}^{\bar{\rho}}, (0)} : CF_*^{[id]}((L, L), H; U) &\rightarrow CF_*^{[id]}((L, L), 0; U) \end{aligned}$$

and their compositions

$$\begin{aligned}\Psi_{\mathcal{H}^\rho, (0)} \circ \Phi_{\mathcal{H}^{\bar{\rho}}, (0)} &: CF_*((L, L), 0; U) \rightarrow CF_*((L, L), H; U), \\ \Phi_{\mathcal{H}^{\bar{\rho}}, (0)} \circ \Psi_{\mathcal{H}^\rho, (0)} &: CF_*((L, L), H; U) \rightarrow CF_*((L, L), 0; U).\end{aligned}$$

Theorem 4.1 and Theorem 6.1 imply that all the above maps properly restrict to the maps between $CF_*^{[id]}(0; U) \cong (C_*(L), \partial_{(0)})$, the singular chain complex, and $CF_*^{[id]}(H; U)$ by isolating the thin trajectories. Since the thin trajectories cannot bubble-off, all these maps become chain maps between them. Therefore $\Psi_{\mathcal{H}^\rho, (0)}$ and $\Phi_{\mathcal{H}^{\bar{\rho}}, (0)}$ induce the isomorphisms between $HF_*^{[id]}((L, L), 0; U) \cong H_*(L)$ and $HF_*^{[id]}((L, L), H; U)$ which are inverses to each other. More precisely, there exist a chain homotopy maps between $\Psi_{\mathcal{H}^\rho, (0)} \circ \Phi_{\mathcal{H}^{\bar{\rho}}, (0)}$ and $id_{C_*(L)}$, and $\Phi_{\mathcal{H}^{\bar{\rho}}, (0)} \circ \Psi_{\mathcal{H}^\rho, (0)}$ and $id_{CF_*(H)}$ respectively. (See [Oh1, FOOO3] for the proof of existence of such a chain homotopy.) Once this is established, we can compute $HF_*^{[id]}((L, L), 0; U)$ inside the cotangent bundle T^*L . Then [Fl4] and [Oh2] prove the theorem.

This finishes the proof. \square

8. LOCALIZATION ON THE COTANGENT BUNDLE

The main purpose of this section is to use the local Floer complex constructed on the cotangent bundle and localize the construction of Lagrangian spectral invariants introduced in [Oh3] which has been further studied in [Oh15].

Although we will not need for the paper [Oh15], we will also localize the triangle product similarly and the basic phase function in the current context of approximations of engulfable topological Hamiltonian loops in Appendix, for a future purpose.

We first specialize the general definition of spectral invariants $\rho^{lag}(F; 1)$ and $\rho_V^{lag}(F; 1_0)$ to the cotangent bundle. In this case, we do not need to use the Novikov ring but only use the coefficient ring R and have only to use the single valued classical action functional

$$\mathcal{A}_F^{cl}(\gamma) = \int \gamma^* \theta - \int_0^1 F(t, \gamma(t)) dt$$

in the evaluation of the level of the chains.

Recall we assume L is connected. Using the isomorphism

$$(\Psi_{\mathcal{H}})_* : H_*(L, \Lambda_{(\omega, L)}) \rightarrow HF_*(F)$$

where $\Psi_{\mathcal{H}}$ is the chain map defined in section 6, we define

$$\rho^{lag}(F; 1) = \inf_{\alpha \in (\Psi_{\mathcal{H}})_*([L])} \lambda_F(\alpha)$$

which is also the same as

$$\inf_{\lambda} \left\{ HF_*^{[id], \lambda}(F) \neq 0 \right\}.$$

This is because $HF_n(F; V)$ or $HF_n(F)$ has rank one and so all isomorphisms $H_*(L) \rightarrow HF_*(F)$ maps the fundamental cycle $[L]$ of L to the same image modulo a non-zero scalar multiple and so the associated spectral invariants coincide (Confomality Axiom [Oh7]). Similarly we define

$$\rho_V^{lag}(F; 1) = \inf_{\alpha \in (\Psi_{\mathcal{H}, (0)})_*([L])} \lambda_F(\alpha)$$

which is also the same as

$$\inf_{\lambda} \left\{ HF_*^{[id], \lambda}(F; V) \neq 0 \right\}.$$

Remark 8.1. We would like to mention that the homomorphism $(\Psi_{\mathcal{H}})_*$ and $(\Psi_{\mathcal{H},(0)})$ do not depend on the choice of homotopy \mathcal{H} . But for the case $(\Psi_{\mathcal{H},(0)})$ the whole family of Hamiltonians $H(s)$ for $s \in [0, 1]$ should be assumed to be V -engulfable. For example, the commonly used the linear homotopy $s \mapsto sF$ may not be V -engulfable even when F is V -engulfable. Because of this, the linear homotopy cannot be used to construct the local chain map in general.

Now we prove the following coincidence theorem of global and local spectral invariants.

Theorem 8.1. *Let $\mathcal{H} = \{H(s)\}$ be an engulfable isotopy with $H(0) = 0$ and $F = H(1)$. Then we have*

$$\rho_V^{lag}(F; 1_0) = \rho^{lag}(F; 1)$$

Proof. For the given family

$$\mathcal{H} : s \mapsto H(s), \quad s \in [0, 1],$$

we consider the continuation of maximal invariant sets defined in section 5. For given one parametric family

$$(J^{para}, H^{para}) \in Map([0, 1]^2, \mathcal{J}_{\omega}) \times C^{\infty}([0, 1]^2 \times M, \mathbb{R})$$

with $H^{para} = \mathcal{H}$ with $H(0) = 0$ and $J^{para} = J_0$, all the Floer trajectories contributing to these maximal invariant sets are thin and so very thin. In particular, the maximal invariant sets \mathcal{S}_s are all contained in the given neighborhood $[0, 1] \times D^{\delta}(T^*L)$ for all of $[0, 1] \times M$ and $\mathcal{S}_0 = L$.

This implies that the local Floer complex $(CF_*^{[id]}(F), \partial_{(0)})$ and the global one $(CF_*(F), \partial)$ define the same complex and also satisfies

$$(\Psi_{\mathcal{H}})_*([L]) = (\Psi_{\mathcal{H},(0)})_*([L])$$

under the identification. This finishes the proof. \square

Now we consider the linear homotopy $s \mapsto sF$ and denote by Ψ_F^{lin} the associated Floer-Piunikhin chain map $C_*(L) \rightarrow CF_*(F)$. It is standard that Ψ_F^{lin} also induces an isomorphism $H_*(L) \rightarrow HF_*(F)$ in global Floer homology. More specifically we have

$$(\Psi_F^{lin})_*([L]) = (\Psi_{\mathcal{H}})_*([L]).$$

However we emphasize that the corresponding cycles $(\Psi_F^{lin})_{\#}([L])$, $(\Psi_{\mathcal{H}})_{\#}([L])$ are different. For example, the general estimate of the level of the cycle $(\Psi_{\mathcal{H}})_{\#}([L])$ involve the derivative $\frac{\partial H(s)}{\partial s}$ which is uncontrolled in the topological Hamiltonian homotopy. On the other hand, using the cycle $(\Psi_F^{lin})_{\#}([L])$, it is easy obtain the upper bound $\rho(F; 1) \leq E^-(F)$ (See [Oh3, Oh6, Oh7]). This together with Theorem 8.1 gives rise to the following inequality

Corollary 8.2. *For any $F \in \mathcal{H}_{\delta}^{engulf}(T^*L; V)$, $\rho_V^{lag}(F; 1_0) \leq E^-(F)$.*

Remark 8.2. We would like to emphasize that unlike the isotopy $\mathcal{H} = \{H(s)\}$ with $H(s) \in \mathcal{H}_\delta^{engulf}(L; V)$, the time-one maps ϕ_{sF}^1 for the linear isotopy $\mathcal{H}^{lin} : s \mapsto sF$ with $F = H(1)$ may not be C^0 -small and hence the associated Floer trajectories of the chain map moduli space could go out of the neighborhood V . Because of this, the linear isotopy cannot be used to define a chain map from $C_*(L)$ to the local Floer complex $CF_*^{[id]}(F; V)$ and so the inequality stated in this corollary does not follow from the standard argument used in [Oh7] to prove $\rho^{lag}(F; 1_0) \leq E^-(F)$ for the global invariant.

9. APPENDIX: LOCAL FLOER COMPLEX OF ENGULFABLE SMOOTH HAMILTONIAN C^0 -APPROXIMATE LOOP

Exposition of this appendix closely follows that of section 4 [Oh6] except that we need to explain the points, if necessary, about why C^0 -smallness of ϕ_F is enough to localize the Floer complex of the fixed point set of ϕ_F^1 .

9.1. Hamiltonian Floer complex. This section reviews the standard construction in Hamiltonian Floer theory. We closely follow exposition of chapter 2 [FOOO2] for some enhancement added which is useful for our purpose later.

Let $\tilde{\mathcal{L}}_0(M)$ be the set of all the pairs $[\gamma, w]$ where γ is a loop $\gamma : S^1 \rightarrow M$ and $w : D^2 \rightarrow M$ a disc with $w|_{\partial D^2} = \gamma$. We identify $[\gamma, w]$ and $[\gamma', w']$ if $\gamma = \gamma'$ and w is homotopic to w' relative to the boundary γ . When a one-periodic Hamiltonian $H : (\mathbb{R}/\mathbb{Z}) \times M \rightarrow \mathbb{R}$ is given, we consider the perturbed functional $\mathcal{A}_H : \tilde{\mathcal{L}}_0(M) \rightarrow \mathbb{R}$ defined by

$$\mathcal{A}_H([\gamma, w]) = - \int w^* \omega - \int H(t, \gamma(t)) dt. \quad (9.1)$$

For a Hamiltonian $H : [0, 1] \times M \rightarrow \mathbb{R}$, we denote its flow, a Hamiltonian isotopy, by $\phi_H : t \mapsto \phi_H^t \in \text{Ham}(M, \omega)$. We denote the time-one map by ϕ_H^1 . We put

$$\text{Fix } \phi_H^1 = \{p \in M \mid \phi_H^1(p) = p\}.$$

Each element $p \in \text{Per}(H)$, the set of 1-periodic orbits, induces a map $z_x = z_x^H : S^1 \rightarrow M$, by the correspondence

$$z_x^H(t) = \phi_H^t(\phi_H^{-1}(x)), \quad (9.2)$$

where $t \in \mathbb{R}/\mathbb{Z} \cong S^1$.

We denote by $\text{Per}(H)$ the set of one-periodic solutions of $\dot{x} = X_H(t, x)$. Then (9.2) provides a one-one correspondence between $\text{Fix } \phi_H^1$ and $\text{Per}(H)$. The set of critical points of \mathcal{A}_H is given by

$$\text{Crit}(\mathcal{A}_H) = \{[z, w] \mid \gamma \in \text{Per}(H), w|_{\partial D^2} = \gamma\}.$$

We consider the universal (downward) Novikov field

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{R}, \lambda_i \rightarrow -\infty \right\}$$

and define a valuation \mathbf{v}_T on Λ by

$$\mathbf{v}_T \left(\sum_{i=1}^{\infty} a_i T^{\lambda_i} \right) = \sup\{\lambda_i \mid a_i \neq 0\}. \quad (9.3)$$

It satisfies the following properties:

$$(1) \quad \mathbf{v}_T(xy) = \mathbf{v}_T(x) + \mathbf{v}_T(y),$$

- (2) $\mathfrak{v}_T(x+y) \leq \max\{\mathfrak{v}_T(x), \mathfrak{v}_T(y)\}$,
- (3) $\mathfrak{v}_T(x) = -\infty$ if and only if $x = 0$,
- (4) $\mathfrak{v}_T(q) = 1$,
- (5) $\mathfrak{v}_T(ax) = \mathfrak{v}_T(x)$ if $a \in R \setminus \{0\}$.

We consider the Λ vector space $\widehat{CF}(H; \Lambda)$ with basis given by the critical point set $\text{Crit}(\mathcal{A}_H)$ of \mathcal{A}_H .

Definition 9.1. We define an equivalence relation \sim on $\widehat{CF}(H; \Lambda)$ so that $[z, w] \sim T^c[z', w']$ if and only if

$$z = z', \quad \int_{D^2} w'^* \omega = \int_{D^2} w^* \omega - c. \quad (9.4)$$

The quotient of $\widehat{CF}(H; \Lambda)$ modded out by this equivalence relation \sim is called the Floer complex of the periodic Hamiltonian H and denoted by $CF(H; \Lambda)$.

Here we do not assume the condition on the Conley-Zehnder indices and work with \mathbb{Z}_2 -grading. In the standard literature on Hamiltonian Floer homology, an additional requirement

$$c_1(\overline{w} \# w') = 0$$

is commonly imposed in the definition Floer complex, denoted by $CF(H)$. For the purpose of the current paper similarly as in [FOOO2], the equivalence relation (9.4) is enough and more favorable in that it makes the associated Novikov ring becomes a field. To differentiate the current definition from $CF_*(H)$, we denote the complex used in the present paper by $CF_*(H; \Lambda)$.

Lemma 9.1. *As a Λ vector space, $CF_*(H; \Lambda)$ is isomorphic to the direct sum $\bigoplus_{\gamma \in \text{Per}(H)} \Lambda$.*

Moreover the following holds: We fix a lifting $[z, w_z] \in \text{Crit}(\mathcal{A}_H)$ for each $z \in \text{Per}(H)$. Then any element x of $CF(M, H; \Lambda)$ is uniquely written as a sum

$$x = \sum_{z \in \text{Per}(H)} x_z [z, w_z], \quad \text{with } x_z \in \Lambda. \quad (9.5)$$

Definition 9.2. (1) Let x be as in (9.5). We define

$$\mathfrak{v}_T(x) = \max\{\mathfrak{v}_T(x_z) + \mathcal{A}_H([z, w_z]) \mid \gamma \in \text{Per}(H)\}.$$

(2) We define a filtration $F^\lambda CF(M, H; \Lambda)$ on $CF(M, H; \Lambda)$ by

$$F^\lambda CF(H; \Lambda) = \{x \in CF(H; \Lambda) \mid \mathfrak{v}_T(x) \leq \lambda\}.$$

We have

$$F^{\lambda_1} CF(H; \Lambda) \subset F^{\lambda_2} CF(H; \Lambda)$$

if $\lambda_1 < \lambda_2$. We also have

$$\bigcap_{\lambda} F^\lambda CF(H; \Lambda) = \{0\}, \quad \bigcup_{\lambda} F^\lambda CF(H; \Lambda) = CF(M; H).$$

(3) We define a metric d_T on $CF(H; \Lambda)$ by

$$d_T(x, x') = e^{\mathfrak{v}_T(x-x')}. \quad (9.6)$$

Then (9.3), (9.4) and Definition 9.2 imply that

$$\mathbf{v}_T(a\mathfrak{x}) = \mathbf{v}_T(a) + \mathbf{v}_T(\mathfrak{x})$$

for $a \in \Lambda^\downarrow$, $\mathfrak{x} \in CF(H; \Lambda)$. We also have

$$T^{\lambda_1} \cdot F^{\lambda_2} CF(H; \Lambda) \subseteq F^{\lambda_1 + \lambda_2} CF(H; \Lambda).$$

Lemma 9.2. (1) \mathbf{v}_T is independent of the choice of the lifting $z \mapsto [z, w_z]$.
 (2) $CF(H; \Lambda^\downarrow)$ is complete with respect to the metric d_T .
 (3) The infinite sum

$$\sum_{[z, w] \in \text{Crit } \mathcal{A}_H} x_{[z, w]}[z, w]$$

converges in $CF(H; \Lambda^\downarrow)$ with respect to the metric d_T if

$$\{[z, w] \in \text{Crit } \mathcal{A}_H \mid \mathbf{v}_T(x_{[z, w]}) + \mathcal{A}_H([z, w]) > -C, x_{[z, w]} \neq 0\}.$$

is finite for any $C \in \mathbb{R}$.

9.2. Isolating local Floer complex. This section is a modification of section 4.1 [Oh6] which treats the case of C^2 -small perturbation of Hamiltonians H following section 3 [Oh2].

As in section 5, we will replace the condition of ϕ_F being C^1 -small by ϕ_F being C^0 -small with the same kind of bound on the Hofer norm $\|F\|$. Once we have established the thick-thin decomposition given in Theorem 4.1, we can safely repeat the arguments laid out in section 4.1 [Oh6], whose summary is now in order.

For given such F , we consider the subset $\mathcal{U} = \mathcal{U}(U_\Delta) \subset \mathcal{L}_0(M)$ of loops given by

$$\mathcal{U} = \{\gamma \in \mathcal{L}_0(M) \mid (\gamma(t), \gamma(0)) \in U_\Delta\}.$$

for a fixed Darboux neighborhood U_Δ of the diagonal $\Delta \subset M \times M$ for all $t \in [0, 1]$. In particular, any periodic orbit z of the flow ϕ_H is contained in $\mathcal{U} \subset \mathcal{L}(M)$ has a canonical isotopy class of contraction w_z . We will always use this convention w_z whenever there is a canonical contraction of z like in this case of small loops. This provides a canonical embedding of $\mathcal{U} \subset \tilde{\mathcal{L}}_0(M)$ defined by

$$z \rightarrow [z, w_z].$$

We denote this canonical embedding by $\mathcal{U}^{[id]}$. This selects a distinguished component of

$$\pi^{-1}(\mathcal{U}) \subset \tilde{\mathcal{L}}_0(M)$$

and other components can be given by

$$\mathcal{U}^{[g]} = g \cdot \mathcal{U}^{[id]}, \quad g \in \Gamma_\omega$$

similarly as before.

Combining the constructions from [Oh6] and section 5, we give

Definition 9.3. Let $J = \{J_t\}$ with $|J_t - J_0|_{C^1} < \varepsilon_3$ with ε_3 sufficiently small. For any $F \in \mathcal{H}_\delta^{engulf}(M)$ and for the given Darboux neighborhood U_Δ of the diagonal $\Delta \subset M \times M$ such that

$$\phi_F^t(\Delta) \subset \text{Int } U_\Delta,$$

we define

$$\mathcal{M}^{[g]}(F, J; \mathcal{U}) = \{u \in \mathcal{M}(F, J) \mid (u(\tau)(t), u(\tau)(0)) \in \text{Int } U_\Delta^{[g]} \text{ for all } \tau\}$$

for each $g \in \Gamma_\omega$. Consider the evaluation map

$$ev : \mathcal{M}(F, J; \mathcal{U}^{[g]}) \rightarrow \mathcal{U} \subset \mathcal{L}_0(M); \quad ev(u) = u(0).$$

For each open neighborhood $U_\Delta \subset M \times M$ of $\Delta \subset U_\Delta$, we define the *local Floer complex* in $\mathcal{U}^{[g]}$ by

$$\mathcal{S}(F, J; \mathcal{U}^{[g]}) := ev(\mathcal{M}(F, J; \mathcal{U}^{[g]})) \subset \mathcal{L}_0(M).$$

We say $\mathcal{S}(F, J; \mathcal{U}^{[g]})$ is *isolated* in $\mathcal{U}^{[g]}$ if its closure is contained in $\mathcal{U}^{[g]}$.

Using Theorem 7.1, we define the local Floer homology, denoted by $HF^{[g]}(F, J; \mathcal{U})$. Furthermore, the pull-back of the action functional \mathcal{A}_F to $\mathcal{U}^{[g]}$ via the above mentioned embedding into $\tilde{\mathcal{L}}_0(M)$ provides a filtration on the local Floer complex $CF^{[g]}(F; \mathcal{U})$.

Therefore by considering the parameterized family

$$\mathcal{S}(G^s, J; \mathcal{U}^{[id]}),$$

the proof of Theorem 7.1 implies that if $G \in \mathcal{H}_\delta^{engulf}(M)$ and δ sufficiently small, $\mathcal{S}(J, G^s : \mathcal{U}^{[id]})$ are isolated in $\mathcal{U}^{[id]}$ for all s and its homology is isomorphic to $H_*(M; R)$. For readers' convenience, we provide the detailed comparison argument between the Hamiltonian Floer complex of $\text{Fix } \phi_G^1$ and the Lagrangian Floer complex of the pair $(\Delta, \text{Graph } \phi_G^1)$ in Appendix borrowing from that of section 4.2 [Oh6].

9.3. Fix ϕ_G^1 versus $\Delta \cap \text{graph } \phi_G^1$. The main goal of this sub-section is to compare the Hamiltonian Floer homology of G with the Lagrangian Floer complex between Δ and graph ϕ_G^1 in the product $(M, \omega) \times (M, -\omega)$ when $G \in \mathcal{H}_\delta^{engulf}(M)$ with δ sufficiently small.

We now compare the local Floer homology $HF^{[id]}(J, G : \mathcal{U})$ of $G \in \mathcal{H}_\delta^{engulf}(M)$ and two versions of its intersection counterparts, one $HF_{J_0 \oplus -J_0, 0}^{[id]}(\text{Graph } \phi_G^1, \Delta : U_\Delta)$ and the other $HF_{(\phi_G)^* J_0 \oplus -J_0, 0 \oplus G}^{[id]}(\Delta, \Delta : U_\Delta)$.

First we note that the two Floer complexes $\mathcal{M}_{J_0 \oplus -J_0, 0}(\text{Graph } \phi_G^1, \Delta : \mathcal{U}^{[id]})$ and $\mathcal{M}_{(\phi_G)^* J_0 \oplus -J_0, 0 \oplus G}(\Delta, \Delta : \mathcal{U}^{[id]})$ are canonically isomorphic by the assignment

$$(\gamma(t), \gamma(t)) \mapsto ((\phi_G^t)^{-1}(\gamma)(t), \gamma(t)).$$

and so the two Lagrangian intersection Floer homology are canonically isomorphic: Here the above two moduli spaces are the solutions sets of the following Cauchy-Riemann equations

$$\begin{cases} \frac{\partial U}{\partial \tau} + (J_0 \oplus -J_0) \frac{\partial U}{\partial t} = 0 \\ U(\tau, 0) \in \text{graph } \phi_G^1, U(\tau, 1) \in \Delta \end{cases}$$

and

$$\begin{cases} \frac{\partial U}{\partial \tau} + ((\phi_G^1)^* J_0) \oplus (-J_0) \left(\frac{\partial U}{\partial t} - X_{0 \oplus G}(U) \right) = 0 \\ U(\tau, 0) \in \Delta, U(\tau, 1) \in \Delta \end{cases}$$

respectively, where $U = (u_1, u_2) : \mathbb{R} \times [0, 1] \rightarrow M \times M$. The relevant action functionals for these cases are given by

$$\mathcal{A}_0([\Gamma, W]) = - \int W^*(\omega \oplus -\omega) \quad (9.7)$$

on $\tilde{\Omega}(\text{Graph } \phi_G^1, \Delta : M \times M)$ and

$$\mathcal{A}_{0 \oplus G}([\Gamma, W]) = \mathcal{A}_0(\Gamma, W) - \int_0^1 (0 \oplus G)(\Gamma(t), t) dt \quad (9.8)$$

on $\tilde{\Omega}(\Delta, \Delta : M \times M)$ where we denote

$\Omega(\text{Graph } \phi_G^1, \Delta : M \times M) = \{\Gamma : [0, 1] \rightarrow M \times M \mid \Gamma(0) \in \text{graph } \phi_G^1, \Gamma(1) \in \Delta, \}$
and similarly for $\Omega(\text{Graph } \phi_G^1, \Delta : M \times M)$. Again the ‘tilde’ means the covering space which can be represented by the set of pairs $[\Gamma, W]$ in a similar way. The relations between the action functionals (9.7), (9.8) and \mathcal{A}_G are evident and respect the filtration under the natural correspondences.

Next we will attempt to compare

$$HF^{[id]}(G, J; \mathcal{U}), \quad HF_{J_0 \oplus -J_0, G \oplus 0}^{[id]}(\Delta, \Delta : U_\Delta).$$

Without loss of any generality, we will concern Hamiltonians G such that $G \equiv 0$ near $t = 0, 1$, which one can always achieve by perturbing G without changing its time-one map.

There is no direct way of identifying the corresponding Floer complexes between the two.

As an intermediate case, we consider the Hamiltonian $G' : M \times [0, 1]$ defined by

$$G'(x, t) = \begin{cases} 2G(x, 2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases},$$

and the assignment

$$(u_0, u_1) \in \mathcal{M}_{J_0 \oplus -J_0, G \oplus 0}^{[id]}(\Delta, \Delta : U_\Delta) \mapsto v \in \mathcal{M}(J, G' : \mathcal{U}^{[id]}) \quad (9.9)$$

with $v(\tau, t) := u_0 \# \bar{u}_1(\tau, t)$. Here the map $u_0 \# \bar{u}_1 : [0, 1] \rightarrow M$ is the map defined by

$$u_0 \# \bar{u}_1(\tau, t) = \begin{cases} u_0(2\tau, 2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ u_1(2\tau, 1 - 2t) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is well-defined and continuous because

$$\begin{aligned} u_0(\tau, 1) &= u_0(\tau, 0) = \bar{u}_1(\tau, 0) \\ \bar{u}_1(\tau, 1) &= u_0(\tau, 1) = u_0(\tau, 0). \end{aligned}$$

Furthermore near $t = 0, 1$, this is smooth (and so holomorphic) by the elliptic regularity since G' is smooth (Recall that we assume that $G \equiv 0$ near $t = 0, 1$). Conversely, any element $v \in \mathcal{M}(J, G' : \mathcal{U}^{[id]})$ can be written as the form of $u_0 \# \bar{u}_1$ which is uniquely determined by v . This proves that (9.9) is a diffeomorphism from $\mathcal{M}_{J_0 \oplus -J_0, G \oplus 0}^{[id]}(\Delta, \Delta : U_\Delta)$ to $\mathcal{M}(J, G' : \mathcal{U}^{[id]})$ which induces a filtration-preserving isomorphism between $HF_{J_0 \oplus -J_0, G \oplus 0}^{[id]}(\Delta, \Delta : U_\Delta)$ and $HF(J, G' : \mathcal{U}^{[id]})$.

Finally, we need to relate $HF(J, G : \mathcal{U}^{[id]})$ and $HF(J, G' : \mathcal{U}^{[id]})$. For this we note that G and G' can be connected by a one-parameter family $G^{para} = \{G^s\}_{0 \leq s \leq 1}$ with

$$G^s(x, t) := \begin{cases} \frac{2}{1+s} G(x, \frac{2}{1+s} t) & \text{for } 0 \leq t \leq \frac{s}{2} \\ 0 & \text{for } \frac{s}{2} \leq t \leq 1. \end{cases}$$

And we have

$$\phi_{G^s}^1 = \phi_G^1 \quad \text{for all } s \in [0, 1].$$

Therefore their spectra coincide, i.e., $\text{Spec}(G) = \text{Spec}(G'^s) = \text{Spec}(G')$. Then there exists an isomorphism

$$h_{G^{para}, J}^{adb} : CF(G' : \mathcal{U}^{[id]}) \rightarrow CF(G : \mathcal{U}^{[id]})$$

respects the filtration and so the induced homomorphism in its homology

$$h_{G^{para}, J}^{adb} : HF(J, G' : \mathcal{U}^{[id]}) \rightarrow HF(J, G : \mathcal{U}^{[id]})$$

becomes a filtration-preserving isomorphism. See [K], [U2], [Oh13] for such a construction.

9.4. Localization of triangle product. A version of localization of triangle product was previously exploited in [Se, Sp, Oh12] for smooth Hamiltonians.

Instead of delving into the localization of triangle product in full generality, we will restrict ourselves to the case of the zero section o_L in the cotangent bundle. Once we isolate the invariant set into a Darboux neighborhood $U \subset M$, we may identify U with a neighborhood V of the zero section $o_L \subset T^*L$ and consider a Hamiltonian F with $\text{supp } F \subset V$. It then follows that due to the non-presence of bubbling effect for the pair (T^*L, o_L) , by an easier argument, we obtain the decomposition $\partial = \partial_{(0)} + \partial'$ of the Floer differential ∂ on $CF_*(F; T^*L)$, and obtain the local Floer complex

$$(CF_*^{[id]}(o_L, F; V), \partial_{(0)}).$$

We first recall the definition of the triangle product described in [Oh4], [FO] and the discussion carried out in section 6 [Oh15]. Similar idea of localizing the triangle product was used in [Se], [Oh12] and [Sp]. Instead of delving into the localization in full generality, we restrict ourselves to the case relevant to our main interest arising from the study in [Oh15].

Let $q \in N$ be given. Consider the Hamiltonians $H : [0, 1] \times T^*N \rightarrow \mathbb{R}$ such that L_H intersects transversely both o_N and T_q^*N . We consider the Floer complexes

$$CF(L_H, o_N), \quad CF(o_N, T_q^*N), \quad CF(L_H, T_q^*N)$$

each of which carries filtration induced from the effective action function given below. We denote by $\mathfrak{v}(\alpha)$ the level of the chain α in any of these complexes.

More precisely, $CF(L_H, o_N)$ is filtered by the effective functional

$$\mathcal{A}^{(1)}(\gamma) := \int \gamma^* \theta + h_H(\gamma(0)),$$

$CF^\mu(o_N, T_q^*N)$ by

$$\mathcal{A}^{(2)}(\gamma) := \int \gamma^* \theta,$$

and $CF(L_H, T_q^*N)$ by

$$\mathcal{A}^{(0)}(\gamma) := \int \gamma^* \theta + h_H(\gamma(0))$$

respectively. We recall the readers that h_H is the potential of L_H and the zero function the potentials of o_N, T_q^*N .

We now consider the triangle product in the chain level, which we denote by

$$\mathfrak{m}_2 : CF(L_H, o_N) \otimes CF(o_N, T_q^*N) \rightarrow CF(L_H, T_q^*N) \quad (9.10)$$

following the general notation from [FOOO1]. This product is defined by considering all triples

$$x_1 \in L_H \cap o_N, x_2 \in o_N \cap T_q^* N, x_0 \in L_H \cap T_q^* N$$

with the polygonal Maslov index $\mu(x_1, x_2; x_0)$ whose associated analytical index, or the virtual dimension of the moduli space

$$\mathcal{M}_3(D^2; x_1, x_2; x_0) := \widetilde{\mathcal{M}}_3(D^2; x_1, x_2; x_0) / PSL(2, \mathbb{R})$$

of J -holomorphic triangles, becomes zero and counting the number of elements thereof.

Definition 9.4. Let $J = J(z)$ be a domain-dependent family of compatible almost complex structures with $z \in D^2$. We define the space $\widetilde{\mathcal{M}}_3(D^2; x_1, x_2; x_0)$ by the pairs $(w, (z_0, z_1, z_2))$ that satisfy the following:

- (1) $w : D^2 \rightarrow T^*N$ is a continuous map satisfying $\bar{\partial}_J w = 0$ on $D^2 \setminus \{z_0, z_1, z_2\}$,
- (2) the marked points $\{z_0, z_1, z_2\} \subset \partial D^2$ with counter-clockwise cyclic order,
- (3) $w(z_1) = x_1$, $w(z_2) = x_2$ and $w(z_0) = x_0$,
- (4) the map w satisfies the Lagrangian boundary condition

$$w(\partial_1 D^2) \subset L_H, w(\partial_2 D^2) \subset o_N, w(\partial_3 D^2) \subset T_q^* N$$

where $\partial_i D^2 \subset \partial D^2$ is the arc segment in between x_i and x_{i+1} ($i \bmod 3$).

We have the following energy estimate

Proposition 9.3 (Proposition 6.2 [Oh15]). *Suppose $w : D^2 \rightarrow T^*N$ be any smooth map with finite energy that satisfy all the conditions given in 9.4, but not necessarily J -holomorphic. We denote by $c_x : [0, 1] \rightarrow T^*N$ the constant path with its value $x \in T^*N$. Then we have*

$$\int w^* \omega_0 = \mathcal{A}^{(1)}(c_{x_1}) + \mathcal{A}^{(2)}(c_{x_2}) - \mathcal{A}^{(0)}(c_{x_0}) \quad (9.11)$$

An immediate corollary of this proposition from the definition of \mathfrak{m}_2 is that the map (9.10) restricts to

$$\mathfrak{m}_2 : CF^\lambda(L_H, o_N) \otimes CF^\mu(o_N, T_q^* N) \rightarrow CF^{\lambda+\mu}(L_H, T_q^* N)$$

and in turn induces the product map

$$*_F : HF^\lambda(L_H, o_N) \otimes HF^\mu(o_N, T_q^* N) \rightarrow HF^{\lambda+\mu}(L_H, T_q^* N) \quad (9.12)$$

in homology. This is because if w is J -holomorphic $\int w^* \omega \geq 0$. This ends the summary of triangle product on the global Floer complex explained in [Oh15].

To localize the above construction to obtain the local analogs

$$\mathfrak{m}_{2,(0)} : CF^\lambda(L_H, o_N; V) \otimes CF^\mu(o_N, T_q^* N; V) \rightarrow CF^{\lambda+\mu}(L_H, T_q^* N; V)$$

and the induced product

$$*_{F,(0)} : HF^\lambda(L_H, o_N; V) \otimes HF^\mu(o_N, T_q^* N; V) \rightarrow HF^{\lambda+\mu}(L_H, T_q^* N; V) \quad (9.13)$$

in homology, we have only to prove the analog to Theorem 4.1 and Theorem 6.1 for the moduli space

$$\widetilde{\mathcal{M}}_3(D^2; x_1, x_2; x_0).$$

Theorem 9.4. *Let V be an open neighborhood of the zero section o_L and let $H \in \mathcal{H}_\delta^{engulf}(T^*L)$. Then for any given open neighborhood V of o_L , there exists some $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$, for any element $w \in \widetilde{\mathcal{M}}_3(L_H, o_N, T_q^*N)$ the following alternative holds:*

- (1) $\text{Image } w \subset V$ and $\max_{z \in \mathbb{R} \times \{0,1\}} d(v(z), o_L) \leq \delta$,
- (2) $\text{Image } w \not\subset V$ and $\int w^* \omega \geq C(J_0, V)$.

Proof. The only difference in the proof of this theorem from Theorem 4.1 and 6.1 is that we also need to use the strong maximum principle along the fiber Lagrangian T_q^*N in addition. We would like to note that the intersection

$$T_q^*N \cap S^\delta(T^*N)$$

is Legendrian and so a J_g -holomorphic curve satisfies strong maximum principle along T_q^*N . We refer to [EHS], [Oh3] for such an application of strong maximum principle to obtain C^0 -estimate. \square

The proof is exactly the same as that of Theorem 4.1 and Theorem 6.1 and so omitted.

We define the ‘thin’ part of \mathbf{m}_2 by counting those elements w from $\widetilde{\mathcal{M}}_3(L_H, o_N, T_q^*N)$ of the type (1) above and decompose

$$\mathbf{m}_2 = \mathbf{m}_{2,(0)} + \mathbf{m}'_2.$$

It also follows that $\mathbf{m}_{2,(0)}$ induces a product map

$$\mathbf{m}_{2,(0)} : CF^\lambda(L_H, o_N; V) \otimes CF^\mu(o_N, T_q^*N; V) \rightarrow CF^{\lambda+\mu}(L_H, T_q^*N; V).$$

It is straightforward to check that this map satisfies

$$\partial_{(0)}(\mathbf{m}_{2,(0)}(x, y)) = \mathbf{m}_{2,(0)}(\partial_{(0)}(x), y) \pm \mathbf{m}_{2,(0)}(x, \partial_{(0)}(y))$$

and so induces a product

$$*_{F,(0)} : HF_*^\lambda(L_H, o_N; V) \otimes HF_*^\mu(o_N, T_q^*N; V) \rightarrow HF_*^{\lambda+\mu}(L_H, T_q^*N; V) \quad (9.14)$$

as in [Oh15].

9.5. Localization of the basic phase function. We consider the Lagrangian pair

$$(o_N, T_q^*N), \quad q \in N$$

and its associated Floer complex $CF(H; o_N, T_q^*N)$ generated by the Hamiltonian trajectory $z : [0, 1] \rightarrow T^*N$ satisfying

$$\dot{z} = X_H(t, z(t)), \quad z(0) \in o_N, z(1) \in T_q^*N. \quad (9.15)$$

Denote by $\text{Chord}(H; o_N, T_q^*N)$ the set of solutions. The differential $\partial_{(H,J)}$ on $CF(H; o_N, T_q^*N)$ is provided by the moduli space of solutions of the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\ u(\tau, 0) \in o_N, u(\tau, 1) \in T_q^*N. \end{cases} \quad (9.16)$$

An element $\alpha \in CF(H; o_N, T_q^*N)$ is expressed as a finite sum

$$\alpha = \sum_{z \in \text{Chord}(H; o_N, T_q^*N)} a_z[z], \quad a_z \in \mathbb{Z}.$$

We denote the level of the chain α by

$$\lambda_H(\alpha) := \max_{z \in \text{supp } \alpha} \{\mathcal{A}_H^{cl}(z)\}. \quad (9.17)$$

The resulting invariant $\rho(H; \{q\})$ is to be defined by the mini-max value

$$f_H(q) := \inf_{\alpha \in [q]} \lambda_H(\alpha)$$

where $[q] \in H_0(\{q\}; \mathbb{Z})$ is a generator of the homology group $H_0(\{q\}; \mathbb{Z})$.

Equivalently, we can consider the pair (L_H, T_q^*N) for the action functional

$$\mathcal{A}^{(0)}(\gamma) := \int \gamma^* \theta + h_H(\gamma(0))$$

defined on $\Omega(L_H, T_q^*N)$ which defines the geometric version of the Floer complex $CF(L_H, T_q^*N)$ via the equation

$$\begin{cases} \frac{\partial v}{\partial \tau} + J_0 \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in L_H, v(\tau, 1) \in T_q^*N. \end{cases} \quad (9.18)$$

Now by the same argument performed in sections 4 and 6, we can localize the Floer complex to $CF(L_H, T_q^*N; V)$ and define the local version of the spectral invariant $\rho_V^{lag}(H; \{q\})$ by

$$f_H^V(q) = \inf_{\alpha \in [q]} \lambda_H(\alpha)$$

where $[q] \in H_0(\{q\}; \mathbb{Z})$ is a generator of the homology group $H_0(\{q\}; \mathbb{Z})$. By varying $q \in N$, this defines a function $f_H^V : N \rightarrow \mathbb{R}$ which is precisely the local version of the basic phase function defined in [Oh3]. We denote the associated graph part of the front W_{R_H} of the L_H by $G_{f_H^V}$.

We summarize the main properties of f_H^V whose proofs are verbatim the same as those established for the (global) basic function f_H in [Oh3], [Oh15] by replacing the global Floer complex $CF_*(H)$ by the local complex $CF_*^{[id]}(H; V)$. First we have

Theorem 9.5. *Let $H = H(t, x) \in \mathcal{H}_\delta^{engulf}(T^*N)$ and the Lagrangian submanifold $L_H = \phi_H^1(o_N)$. Consider the function f_H^V defined above. Then for any $x \in L_H$*

$$f_H^V(\pi(x)) = h_H(x) = \mathcal{A}_H^{cl}(z_x^H) \quad (9.19)$$

*for some Hamiltonian chord z_x^H ending at $L_H \cap T_{\pi(x)}^*N$.*

Once we have achieved localizations of various entities arising in Floer complex in the previous subsection, the following equality can be proven by the same argument used in the proof of Theorem 8.1 using the localized version of Lagrangian spectral invariants and basic phase function. We omit the details of its proof.

Theorem 9.6. *Let $V \subset T^*N$ be as before. Then*

$$f_H^V = f_H$$

for any V -engulfable H .

An immediate corollary of this theorem is the following inequality.

Corollary 9.7. *For any Hamiltonian $H \in \mathcal{H}_\delta^{engulf}(T^*N)$,*

$$\max f_H^V \leq E^-(H). \quad (9.20)$$

*Furthermore if $H, H' \in \mathcal{H}_\delta^{engulf}(T^*N)$*

$$\|f_H^V - f_{H'}^V\|_\infty \leq \|H - H'\|. \quad (9.21)$$

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